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# Non-commutative generalization of some probabilistic results from representation theory

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The subject of this thesis is the non-commutative generalization of some probabilistic results that occur in representation theory. The results of the thesis are divided in three parts, which are summarized here.

Weingarten calculus and free easy quantum groups: Easy quantum groups have been defined in [2] as a class of orthogonal compact quantum groups whose associated intertwiners are described by set partitions. This class of compact quantum groups contains important examples of quantum groups as the classical orthogonal and symmetric groups and their free analog, the free orthogonal and free symmetric groups (see [11,12]). It has been later possible to systematically develop the Weingarten calculus on these compact quantum groups in order to get some probabilistic results: in particular, they recovered the convergence results of Diaconis and Shahshahani (see [4]) on the orthogonal and symmetric group, and extended them to the free case. The usual gaussian and Poisson laws are replaced in the free case by the semicircular and the Marchenko-Pastur laws, their free analog in free probability theory.
The first part of the thesis is devoted to the generalization of this framework in the unitary case. Namely the compact quantum groups are not assumed to be orthogonal anymore, but their intertwiner spaces are still described by set partitions with colors. The classical example is given by the classical unitary group whose intertwiner spaces are described by permutations (which can be seen as two-colored pair partitions) through the Schur-Weyl duality. We classify all unitary easy quantum groups whose intertwiner spaces are described by non-crossing partitions, and develop the Weingarten calculus on these quantum groups. As an application of the previous work, we recover the results of Diaconis and Shahshahani on the unitary group and extend those results to the free unitary group.

Free wreath product: The free wreath product is a non-commutative analog of the classical wreath product. The free wreath product is an algebraic construction that produces a new compact quantum group from a compact quantum group and a non-commutative permutation group. This construction arises naturally in the study of quantum symmetries of lexicographical products of graphs. In the classical case, the representation theory of a wreath product is well-know (see for example [8], Part 1, Annex B) and the Haar measure has a straightforward expression. It is for example easy to prove that the fundamental character of a wreath product with the symmetric group $S_n$ converges toward a compound Poisson law as n goes to infinity. However in the free case, the Haar state doesn't have any straightforward expression. For instance Banica and Bichon conjectured in [1] that in some cases, the fundamental character of a free wreath product is distributed as the free multiplicative convolution of the law of the two initial fundamental characters.

In the second part of the thesis, we study the free wreath product. First, we more specifically study the free wreath product with the free symmetric group by giving a description of the intertwiner spaces: several probabilistic results are deduced from this description. Then, we relate the intertwiner spaces of a free wreath product with the free product of planar algebras, an object which has been defined by Bisch and Jones in [6]. This relation allows us to express the law of the character of a free wreath product as a free multiplicative convolution of the initial laws, which proving the conjecture of Banica and Bichon.

<u>Martin boundary of the Zig-zag lattice:</u> The ring QSym of quasi-symmetric functions is a refinement of the ring of symmetric functions, in the sense that any symmetric function has a decomposition in terms of quasi-symmetric ones. An important basis of this ring is called the fundamental basis, and its elements have a monomial expansion similar to the Schur basis of the ring of symmetric functions: this expansion is indexed by semi-standard filling of ribbon Young diagrams for the fundamental basis of QSym and by semi-standard filling of Young diagram for the Schur basis of Sym. The multiplication structure of the Schur basis is encoded by an important graph which is called the Young graph and denoted by Y. This graph has many applications in the representation theory of the infinite group $S_\infty$ and in the probabilistic behavior of some discrete processes. It has been intensively studied by Thoma, Vershik and Kerov in [9, 10, 7]. In particular they identified the minimal and Martin boundaries of Y , and proved that the two coincide. The analog of Y for the fundamental basis of QSym is the graph Z of Zigzag diagrams. This lattice has been deeply studied by Gnedin and Olshanski who identified in [5] its minimal boundary. They conjectured that the minimal and Martin boundaries also coincide on Z.

In the last part of the thesis, we prove that the minimal and the Martin boundaries of Z are the same. In order to prove this, we give some precise estimates on the uniform standard filling of a large ribbon Young diagram: we prove that in a uniform filling, the filling of distant cells become independent in a certain sense. This yields a positive answer to a conjecture that Bender, Helton and Richmond gave in [3].

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